Primitive spatial graphs and graph minors

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Definition

G : graph
$\phi : G \to S^3$: embedding

$\boxed{\phi \text{ is free}}$

$\iff$ $\pi_1(S^3 - \phi(G))$ is a free group

$\boxed{\phi \text{ is flat}}$

$\iff$ $\forall$ cycle $C \subset G$, $\exists$ disk $D \subset S^3$,
s.t. $D \cap \phi(G) = \partial D = \phi(C)$

$\boxed{\phi \text{ is primitive}}$

$\iff$ $\forall$ component G_i of G,
$\forall$ spanning tree T_i of G_i,
the bouquet $\phi(G_i)/\phi(T_i)$ is trivial

Fundamental Theorem and Conjecture

Theorem (Robertson-Seymour-Thomas)

ϕ is flat $\iff$
$\forall\ H \subset G$, $\phi|_H$ is free

Theorem 1

ϕ is primitive $\iff$
$\forall$ **connected** $H \subset G$, $\phi|_H$ is free

Theorem (Robertson-Seymour-Thomas)

G is flat $\iff$ G is linkless

Conjecture 1

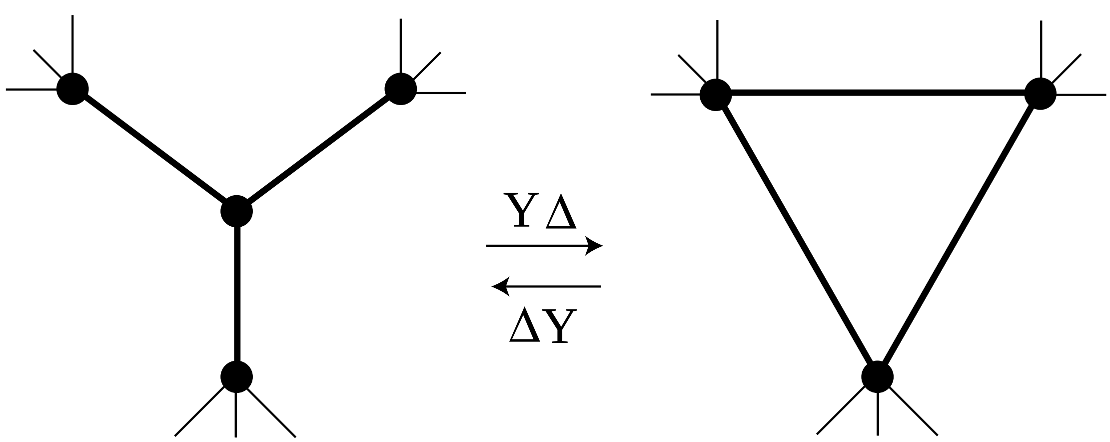
G is primitive $\iff$ G is knotless

Theorem 2

ϕ is primitive $\iff$
both of ϕ_{G-e} and $\phi_{G/e}$ are primitive

Theorem 3

Suppose $\phi(H) \xrightarrow{\Delta Y} \phi'(G)$,
where the 3-cycle in $\phi(H)$ bounds a disk.
Then
$\phi(H)$ is primitive $\iff$ $\phi'(G)$ is primitive



$Y\Delta$- and ΔY-exchange

Remark

Theorem 2 and 3 also hold on knotless.

Graph minor

Theorem 4

"Primitive" $\mathcal{P}$ is a minor-closed property.

Theorem 5

$\Omega(\mathcal{P}) \supset (K_7\text{-family}) \cup (K_{3,3,1,1}\text{-family})$

<u>Remark</u>

$\exists\ G \in \Omega(\mathcal{P}) - (K_7\text{-family}) \cup (K_{3,3,1,1}\text{-family})$

<u>Proof of Remark</u>

Foisy graph F is intrinsically knotted.

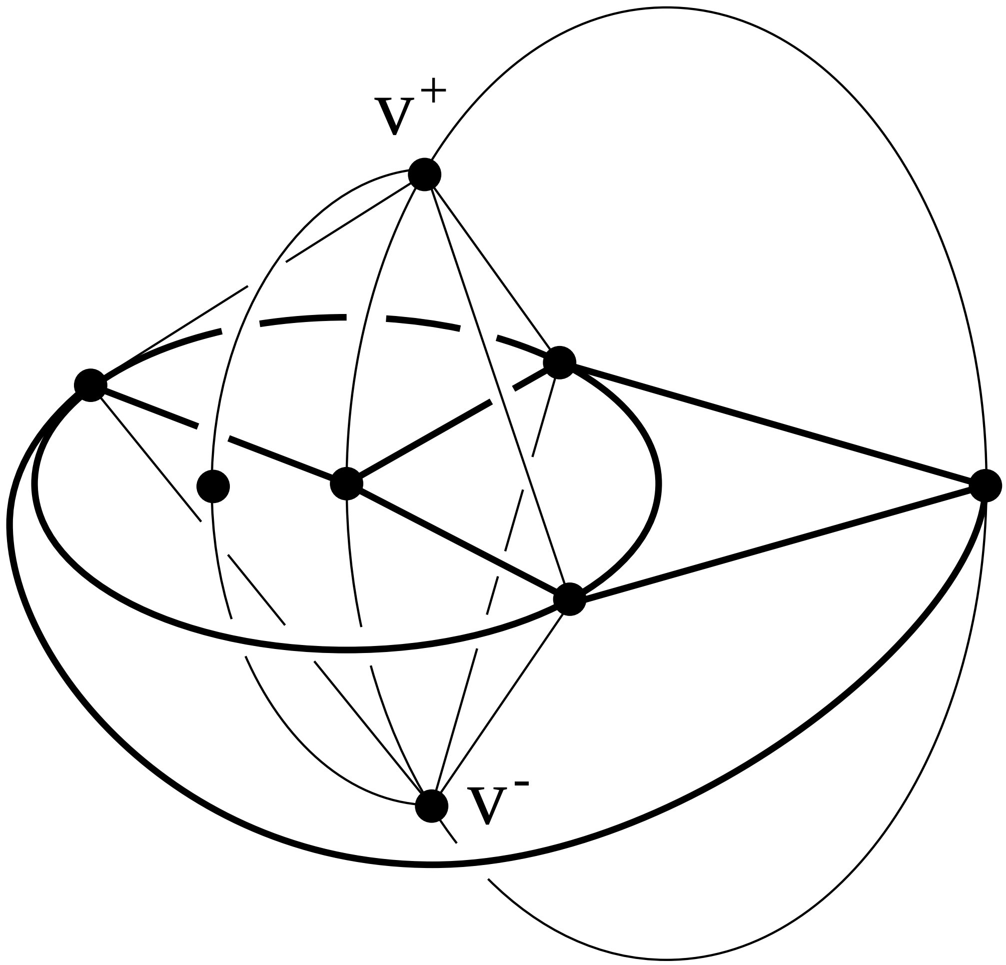
$\Rightarrow$ F is not primitive.

$\Rightarrow \exists\ G \in \Omega(\mathcal{P})$ s.t. $G \prec F$

$\Rightarrow G \notin (K_7\text{-family}) \cup (K_{3,3,1,1}\text{-family})$

Theorem 6

(planar graph) $*\,(v^{+}, v^{-})$ is primitive.



$K_7 - e$ is a minor of planar$*(v^{+}, v^{-})$

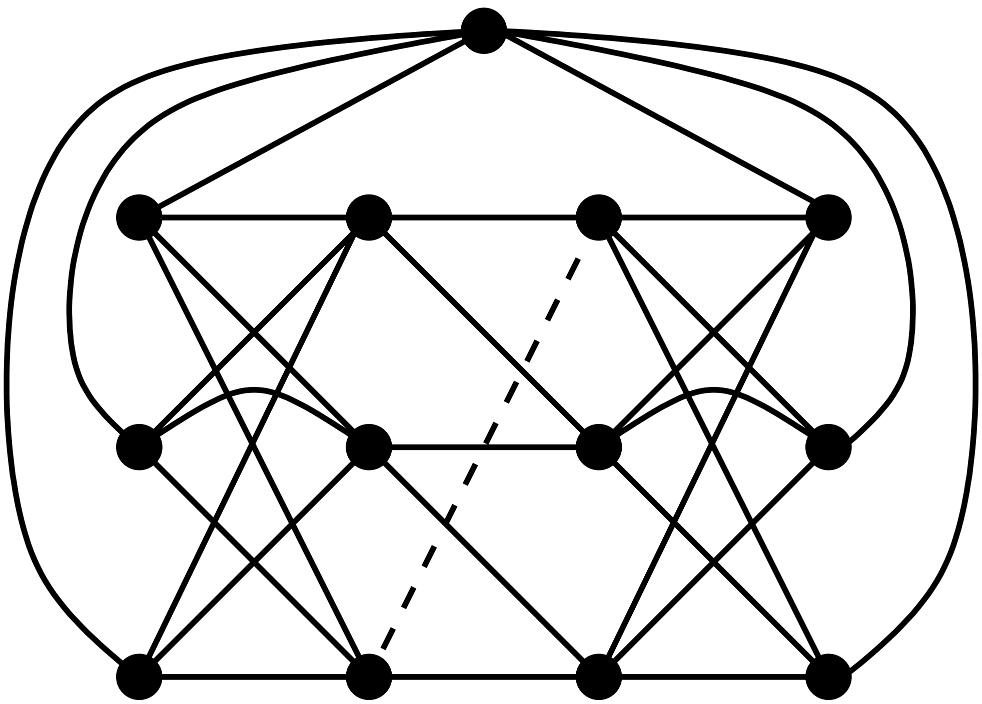
Proposition 1

F : Foisy graph

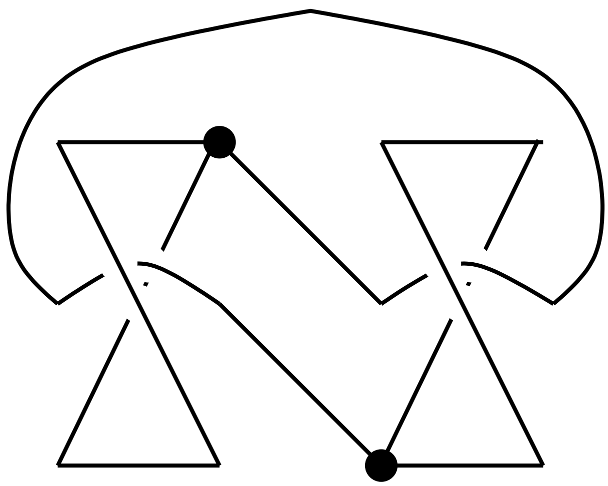
$\widehat{F}'$: the regular projection of $F' = F - e$

Then

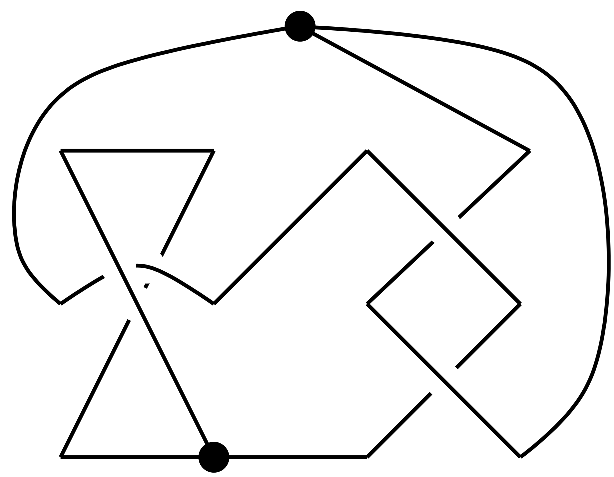
any spatial embedding of F' obtained from $\widehat{F}'$ contains a non-free handcuff graph.



(1): $\widehat{F}'$



(2.1)



(2.2)

Non-free handcuff graphs included in $\phi(F')$

Problem 1

Any embedding of F' contains a non-trivial knot or a non-free Handcuff graph.

Remark

If Problem 1 is true, then $\Omega(\mathcal{P}) \neq \Omega(\mathcal{KL})$.

Primitive embedding

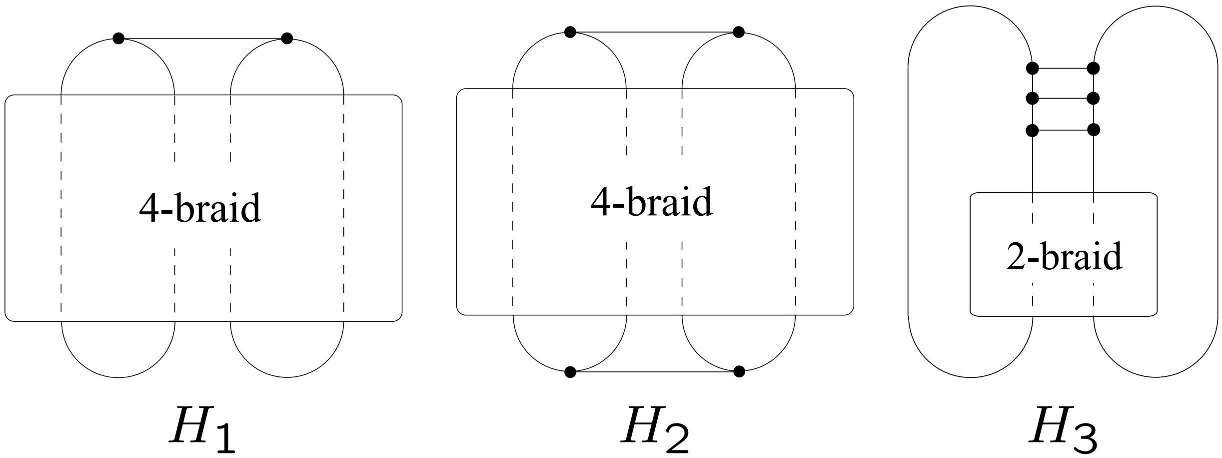
Theorem 7

If G has no disjoint cycles, then
ϕ is primitive $\iff$ ϕ is flat

Theorem 8

Any primitive embedding of H_n forms :

1. a 2-bridge link with an upper tunnel if $n = 1$.
2. a 2-bridge link with an upper tunnel and a lower tunnel if $n = 2$.
3. a $(2, q)$-torus link with three parallel tunnels if $n = 3$.



Primitive embedding of H_n ($n = 1, 2, 3$)

Theorem 9

An n-component link contained in a primitive embedding of a connected graph has bridge number n.

Conjecture 2

Primitive embeddings of a 5-connected graph are unique up to reflections.

Theorem 10

A planar graph has a unique primitive embedding if and only if it has no disjoint cycles.

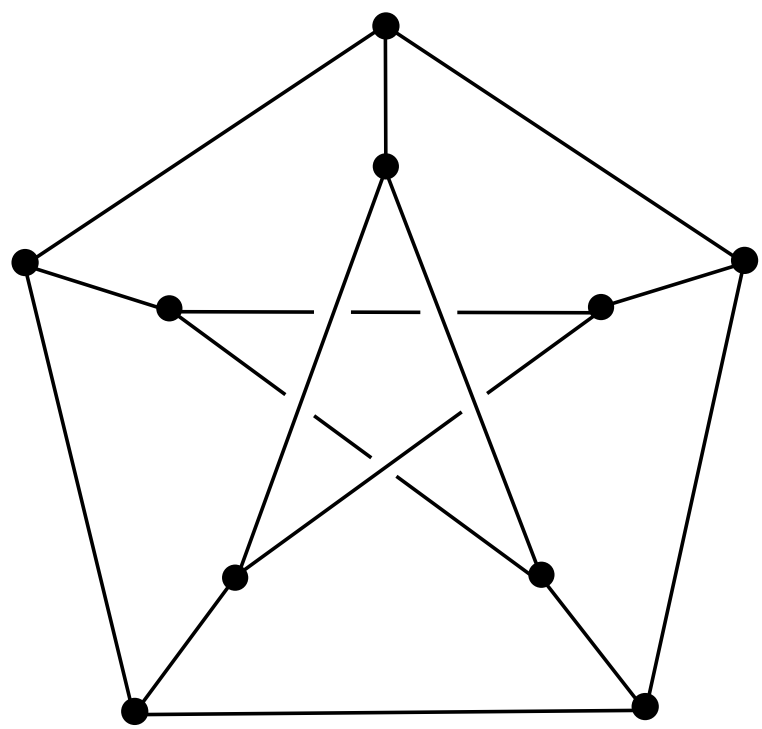
Moreover, if a planar graph has disjoint cycles, then it has infinitely many primitive embeddings.

Theorem 11

Let G be a graph in the Petersen family. Then for any link contained in a primitive embedding of G is either the trivial link or the Hopf link.

Theorem 12

The Petersen graph has a unique primitive embedding.



Conjecture 3

Any graph in the Petersen family has a unique primitive embedding.

Proof

Lemma 1

$\mathcal{C}$: a property preserved under taking minors, multiplication of edges, adding loops, and $Y\Delta$-exchanges.

H : a graph obtained from G by a ΔY-exchange.

Suppose that G does not have $\mathcal{C}$ and suppose that H is a forbidden graph for $\mathcal{C}$.

Then G is also a forbidden graph for $\mathcal{C}$.

Proof of Theorem 5

K_7-family and $K_{3,3,1,1}$-family are not primitive since they are intrinsically knotted.

K_7-family and $K_{3,3,1,1}$-family are obtained from terminal graphs H_{12} and C_{14} in K_7-family and Q_2, Q_3 and R_1 in $K_{3,3,1,1}$-family by $Y\Delta$-exchanges.

Let G be one of these terminal graphs.

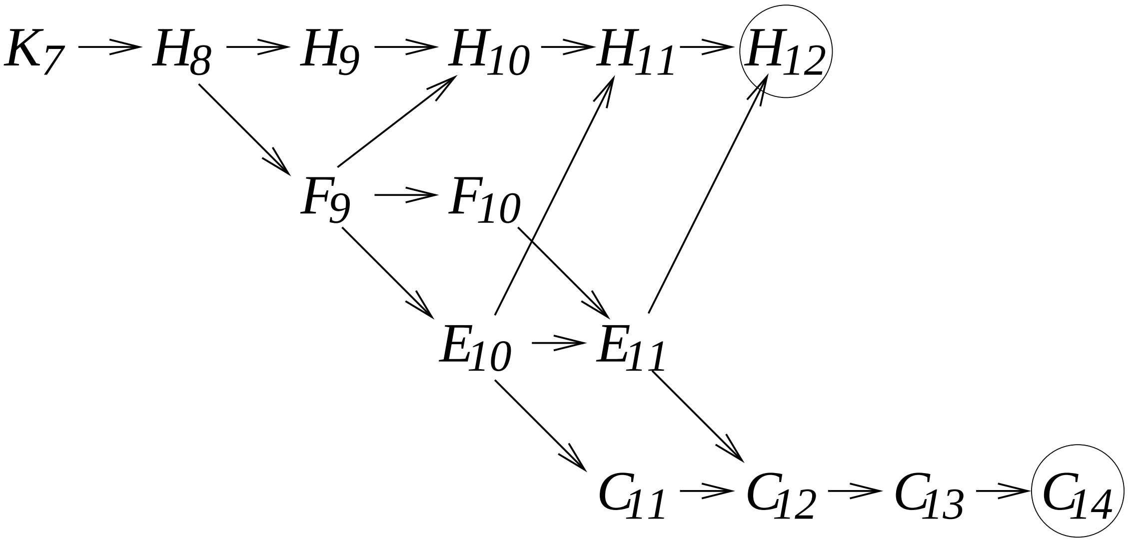
It can be checked that for any edge e, $G-e$ and G/e are planar graphs joined with two vertices.

By Theorem 6, $G-e$ and G/e are primitive, hence G is a forbidden graph for $\mathcal{P}$.

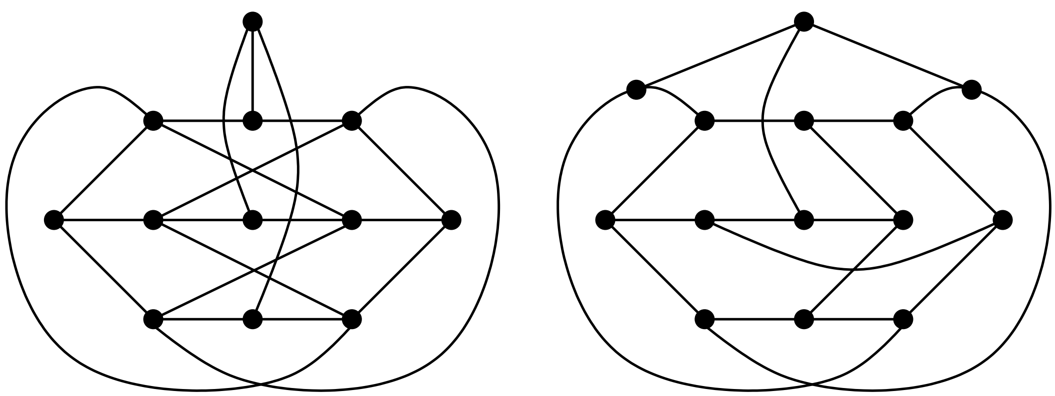
We note that $\mathcal{P}$ is preserved under taking minors, multiplication of edges, adding loops, and $Y\Delta$-exchanges.

Now, by Lemma 1, all graphs in K_7-family and $K_{3,3,1,1}$-family are forbidden graphs for $\mathcal{P}$.

Terminal graphs

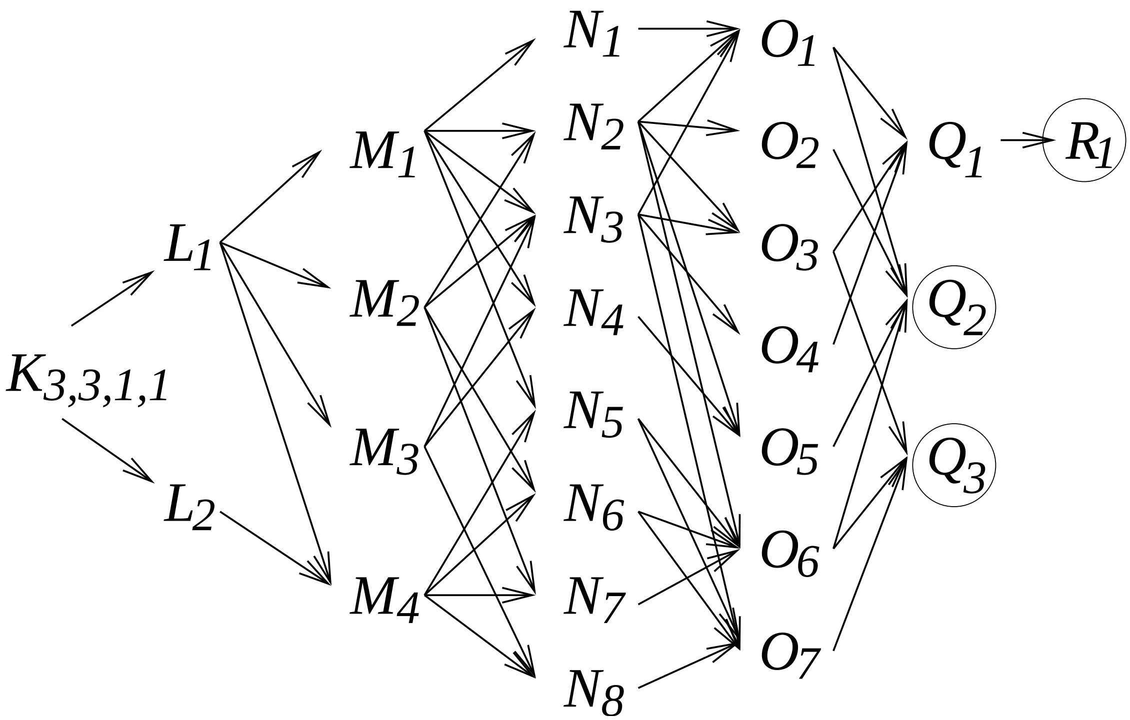


K_7-family

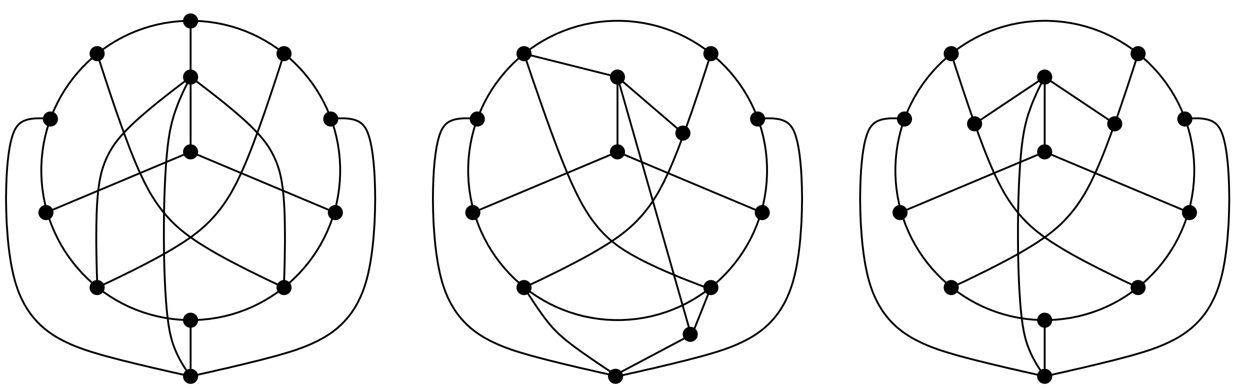


H_{12} C_{14}

Terminal graphs of the K_7-family



$K_{3,3,1,1}$-family



Q_2 Q_3 R_1

Terminal graphs of the $K_{3,3,1,1}$-family