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UNIQUENESS OF ESSENTIAL FREE TANGLE DECOMPOSITIONS OF KNOTS AND LINKS

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ABSTRACT. In this paper, we show that if a knot admits an essential free 2-string tangle decomposition, then essential 2-string tangle decompositions of the knot are unique up to isotopy, and we characterize the link types which admit non-isotopic essential free 2-string tangle decompositions.

1. INTRODUCTION

Let B be a 3-ball and $t = t_1 \cup \ldots \cup t_n$ a union of mutually disjoint n arcs properly embedded in B. Then we call the pair (B, t) an *n-string tangle*. We say that an n-string tangle (B, t) is *trivial* if (B, t) is homeomorphic to $(D \times I, \{x_1, \ldots, x_n\} \times I)$ as pairs, where D is a 2-disk and x_i is a point in $intD$ $(i = 1, \ldots, n)$. According to [1], we say that (B, t) is *essential* if $cl(\partial B - N(t))$ is incompressible and boundary-incompressible in $cl(B - N(t))$. And, according to [3], we say that (B, t) is *free* if $\pi_1(B - t)$ is a free group. We note that (B, t) is free if and only if $cl(B - N(t))$ is a handlebody ([2, 5.2]).

Let L be a knot or link in S^3, and let (B, t) and (B', t') be n-string tangles. We say that a union $(B, t) \cup (B', t')$ is an *n-string tangle decomposition* of L if $S^3 = B \cup B', B \cap B' = \partial B = \partial B', \partial t = \partial t'$ and $L = t \cup t'$. We say that an n-string tangle decomposition $(B, t) \cup (B', t')$ of L is *essential* (*free* resp.) if both (B, t) and (B', t') are essential (free resp.). Let $(B, t) \cup (B', t')$ and $(C, s) \cup (C', s')$ be n-string tangle decompositions of L. Then we say that these tangle decompositions are *mutually isotopic* if there is an ambient isotopy $\{f_t\} : S^3 \to S^3$ $(t \in [0, 1])$ such that $f_0 = id$, $f_1(\partial B) = \partial C$ and $f_t(L) = L$ for any $t \in [0, 1]$.

Then our result is ;

Theorem 1.1. *Let L be a knot or link in S^3 which admits an essential free 2-string tangle decomposition. Then L admits non-isotopic essential 2-string tangle decompositions if and only if L is equivalent to a 2-component Montesinos link*

$M(0;(\alpha_1,\beta_1),(2,1),(\alpha_2,\beta_2),(2,1))$ illustrated in Figure 1, where α_i and β_i are coprime integers, and α_i is an odd integer greater than 1 $(i = 1, 2)$.

Moreover, if L is the Montesinos link, then L admits exactly two essential free 2-string tangle decompositions up to isotopy, and any essential 2-string tangle decomposition of L is isotopic to one of those two.

Corollary 1.2. *If a knot K admits an essential free 2-string tangle decomposition, then essential 2-string tangle decompositions of K are unique up to isotopy.*

Remark 1.3. *The two essential free 2-string tangle decompositions of the Montesinos link in Theorem 1.1 are given by the 2-spheres P and Q indicated in Figure 1.*

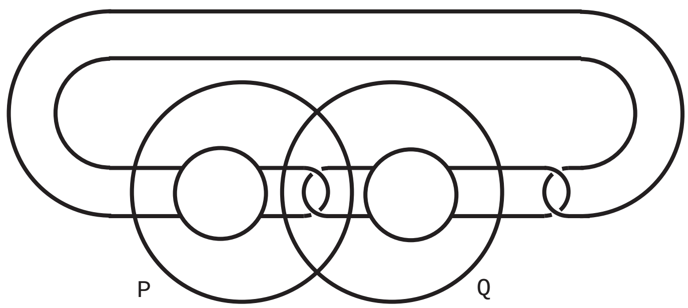

FIGURE 1. $M(0;(\alpha_1,\beta_1),(2,1),(\alpha_2,\beta_2),(2,1))$

Throughout this paper, we work in the piecewise linear category. For an n-manifold X and a subcomplex Y of X, $N(Y)$ or $N(Y;X)$ will denote a regular neighborhood of Y in X.

2. Natures of free tangles

In this section, we prepare some lemmas for Theorem 1.1.

Lemma 2.1. *Let V be a handlebody, F a separating surface properly embedded in V, and V_1 and V_2 the closure of the components of $V - F$. If F is incompressible in V, then both V_1 and V_2 are handlebodies.*

Proof. Since F is incompressible and two-sided in V, both homomorphisms $\pi_1(F) \to \pi_1(V_1)$ and $\pi_1(F) \to \pi_1(V_2)$ induced by the inclusion maps are injective. Therefore both homomorphisms $\pi_1(V_1) \to \pi_1(V)$ and $\pi_1(V_2) \to \pi_1(V)$ induced by the inclusion maps are injective. Thus both $\pi_1(V_1)$ and $\pi_1(V_2)$ are subgroups of $\pi_1(V)$, hence free groups. Then by [2, 5.2], both V_1 and V_2 are handlebodies. □

The following Lemmas 2.2, 2.3 and 2.4 follow Lemma 2.1.

Lemma 2.2. *Let (B, t) be a free 2-string tangle and D a disk properly embedded in B which intersects t in a single point. Then D is isotopic $rel.t$ to a disk in ∂B.*

Lemma 2.3. *Let (B, t) be a free 2-string tangle. If (B, t) is inessential, then (B, t) is trivial.*

Lemma 2.4. *Let $(B, t_1 \cup t_2)$ be a free 2-string tangle and A an annulus properly embedded in $B - (t_1 \cup t_2)$ which does not separate t_1 and t_2. If A is incompressible in $B - (t_1 \cup t_2)$, then A is isotopic $rel.t_1 \cup t_2$ to an annulus in $\partial B - t$.*

The following Lemma 2.5 will be used for Lemmas 2.6 and 2.8.

Lemma 2.5. *(*[5, Proposition 1.6] , [1, Proposition 2.1]*) Let M be an orientable closed 3-manifold with a genus two Heegaard splitting (V_1, V_2). If M contains a 2-sphere S such that each component of $S \cap V_1$ is a non-separating disk in V_1 and $S \cap V_2$ is incompressible and not ∂-parallel in V_2, then M has a lens space or $S^2 \times S^1$ summand.*

Lemma 2.6. *Let (B, t) be a free 2-string tangle and S a 2-sphere in $intB$ which intersects t in four points. If $S - t$ is incompressible in $B - t$, then S is isotopic $rel.t$ to ∂B.*

Proof. Suppose S is not isotopic $rel.t$ to ∂B for a contradiction. Glue a 3-ball B' to B along their boundaries. Put $V_1 = B' \cup N(t; B)$ and $V_2 = cl(B - N(t; B))$. Then, (V_1, V_2) is a genus two Heegaard splitting of the 3-sphere $B \cup B'$, each component of $S \cap V_1$ is a non-separating disk in V_1, and $S \cap V_2$ is incompressible and not ∂-parallel in V_2. In consequence of this observations and Lemma 2.5, the 3-sphere $B \cup B'$ must have a lens space or $S^2 \times S^1$ summand. Thus we obtain a contradiction. □

The following Lemma 2.7 will be used for Lemma 2.8.

Lemma 2.7. *(*[4, Theorem 0.1]*) Let L be a tunnel number one link. Then L is composite if and only if L is a connected sum of a 2-bridge knot and a Hopf link. Moreover, any unknotting tunnel γ for L is isotopic to an arc obtained from the upper or lower tunnel for the 2-bridge knot (Figure 2).*

Now we define a *Montesinos tangle* as a "partial sum" of n rational tangles of slope $\beta_i/\alpha_i (i = 1, \dots, n)$ illustrated in Figure 3, and denote it by $T(\beta_1/\alpha_1, \dots, \beta_n/\alpha_n)$.

Lemma 2.8. *Let $(B, t_1 \cup t_2)$ be a free 2-string tangle and D a disk properly embedded in B which intersects $t_1 \cup t_2$ in two points. If $D - (t_1 \cup t_2)$ is incompressible in $B - (t_1 \cup t_2)$ and D is not isotopic $rel.t_1 \cup t_2$ to a disk in ∂B, then $(B, t_1 \cup t_2)$ is homeomorphic $rel.\partial B$ to a Montesinos tangle $T(\beta/\alpha, 1/2)$ as pairs, where α and β are coprime integers and α is an odd integer greater than 1 (Figure 4). In addition, $(B, t_1 \cup t_2)$ is essential.*

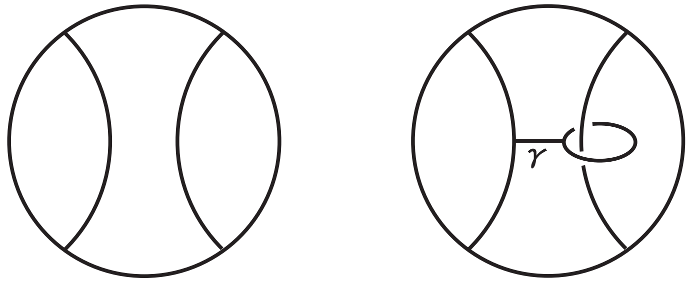

FIGURE 2. 2-bridge decomposition of the 2-bridge knot

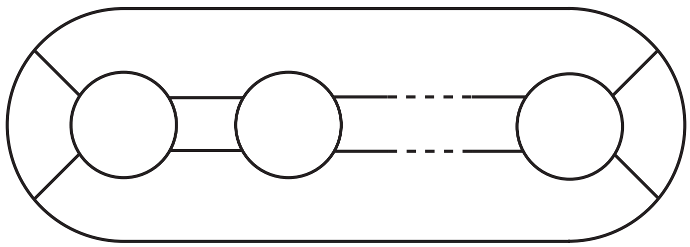

FIGURE 3. Montesinos tangle $T(\beta_1/\alpha_1, \dots, \beta_n/\alpha_n)$

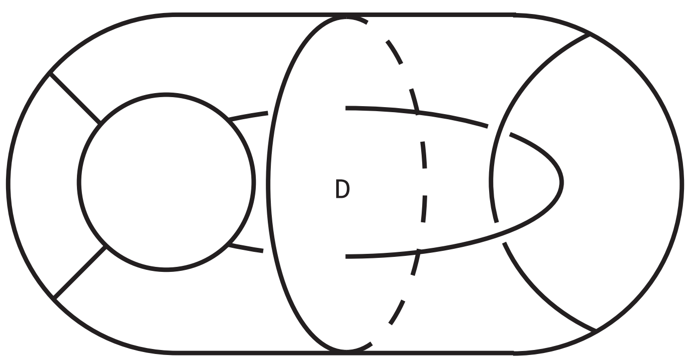

FIGURE 4. Montesinos tangle $T(\beta/\alpha, 1/2)$

Proof. Since $D-(t_1\cup t_2)$ is incompressible in $B-(t_1\cup t_2)$ and D intersects $t_1\cup t_2$ in two points, ∂D splits the four points $\partial t_1\cup\partial t_2$ in ∂B into pairs of two points. Thus D separates $(B, t_1\cup t_2)$ into two 2-string tangles, and we denote them by $(B_1, t_1^1\cup t_2^1)$ and $(B_2, t_1^2\cup t_2^2)$. Here, note that by Lemma 2.1, $(B_i, t_1^i\cup t_2^i)$ is free $(i=1,2)$.

Claim 2.9. $(B_i, t_1^i\cup t_2^i)$ *is trivial* $(i=1,2)$.

Proof. Suppose $\partial B_i-(t_1^i\cup t_2^i)$ is incompressible in $B_i-(t_1^i\cup t_2^i)$. Let S^i be a 2-sphere in $intB$ which is obtained by pushing in ∂B_i into $intB$ slightly. Then $S_i-(t_1\cup t_2)$ is incompressible in $B-(t_1\cup t_2)$ because $D-(t_1\cup t_2)$ is incompressible in $B-(t_1\cup t_2)$. Hence by Lemma 2.6, S_i is isotopic $rel.t_1\cup t_2$ to ∂B. This implies that D is isotopic $rel.t_1\cup t_2$ to a disk in ∂B, and contradicts the hypothesis of the Lemma. Consequently, $\partial B_i-(t_1^i\cup t_2^i)$ is compressible in $B_i-(t_1^i\cup t_2^i)$. Hence by Lemma 2.3, $(B_i, t_1^i\cup t_2^i)$ is trivial. □

By Claim 2.9, we may assume that $(B, t_1\cup t_2)$ is a Montesinos tangle $T(\beta_1/\alpha_1, \beta_2/\alpha_2)$ and $(B_i, t_1^i\cup t_2^i)$ is a rational tangle of slope β_i/α_i $(i=1,2)$

Claim 2.10. *D intersects only one component of $t_1\cup t_2$.*

Proof. Suppose D intersects both components of $t_1\cup t_2$. Let B' be a 3-ball and D' a disk properly embedded in B'. Glue B' to B so that $\partial B'=\partial B$ and $\partial D'=\partial D$. Put $V_1=B'\cup N(t_1\cup t_2;B)$, $V_2=cl(B-N(t_1\cup t_2;B))$ and $S=D\cup D'$. Then, (V_1,V_2) is a genus two Heegaard splitting of the 3-sphere $B\cup B'$, each component of $S\cap V_1$ is a non-separating disk in V_1, and $S\cap V_2$ is incompressible and not ∂-parallel in V_2. In consequence of this observations and Lemma 2.5, the 3-sphere $B\cup B'$ must have a lens space or $S^2\times S^1$ summand. This is absurd. □

By Claim 2.10, we may assume that $D\cap(t_1\cup t_2)=D\cap t_1$, $t_1^1\cup t_2^1=t_1\cap B_1$, $t_1^2=t_1\cap B_2$ and $t_2^2=t_2$. In addition, α_1 is an odd integer greater than 1 because $(B, t_1\cup t_2)$ is a 2-string tangle and D is not isotopic $rel.t_1\cup t_2$ to a disk in ∂B.

Let $(C, s_1\cup s_2)$ be a trivial 2-string tangle, E a disk properly embedded in C which separates the two strings $s_1\cup s_2$, and γ a "trivial" arc which connects s_1 and s_2 (Figure 5).

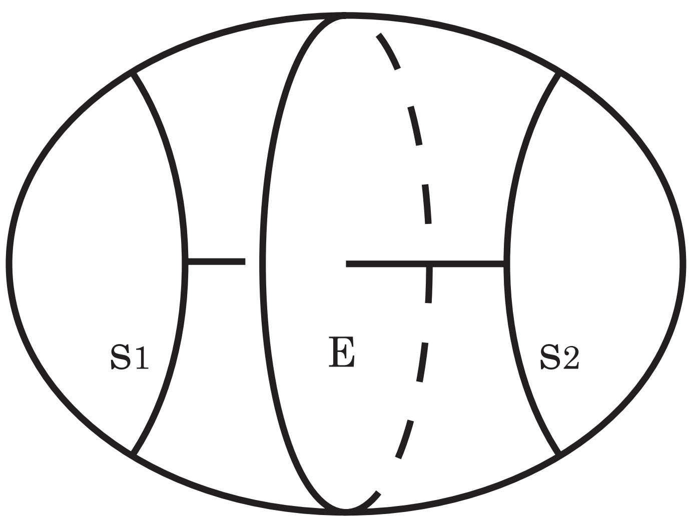

FIGURE 5. $(C, s_1\cup s_2)$ with γ

Let L be the link obtained from $(B, t_1 \cup t_2)$ by attaching $(C, s_1 \cup s_2)$ so that $\partial t_i = \partial s_i (i = 1, 2)$ and $\partial D = \partial E$. Then, since $cl((B \cup C) - N(L \cup \gamma)) = cl(B - N(t_1 \cup t_2)) \cup cl(C - N(s_1 \cup s_2 \cup \gamma)) \cong cl(B - N(t_1 \cup t_2))$ is a genus two handlebody, L is a tunnel number one link with an unknotting tunnel γ.

Claim 2.11. *L is a composite link with the decomposing sphere $D \cup E$.*

Proof. Let C_1 and C_2 be the closure of the components of $C - E$ such that $C_i \supset s_i$ $(i = 1, 2)$. Since α_1 is an odd integer greater than 1, $(B_1 \cup C_1, t_1^1 \cup t_2^1 \cup s_1)$ is a non-trivial 1-string tangle. This completes the proof because $(B_2 \cup C_2, t_1^2 \cup t_2^2 \cup s_2)$ is not a trivial 1-string tangle. □

By Claim 2.11 and Lemma 2.7, L is a connected sum of the 2-bridge knot $t_1 \cup s_1$ and the Hopf link $t_2 \cup s_2$. Then by rotating $intB$ in a "horizontal" axis if necessary, we may assume that $(B, t_1 \cup t_2)$ is the Montesinos tangle in the Lemma.

Finally, if $(B, t_1 \cup t_2)$ is inessential, then by Lemma 2.3, it is trivial, and hence (B, t_1) is a trivial 1-string tangle. On the other hand, since α_1 is an odd integer greater than 1 and (B_2, t_2^1) is a trivial 1-string tangle, (B, t_1) is a non-trivial 1-string tangle. This completes the proof of the Lemma. □

3. Proof of Theorem 1.1

Proof. Let L be a knot or link in S^3 with an essential free 2-string tangle decomposition $(B_1, t_1) \cup (B_2, t_2)$. Suppose that L admits another essential tangle decomposition $(C_1, s_1) \cup (C_2, s_2)$ which is not isotopic to the above one. Put $P = \partial B$ and $Q = \partial C$.

If $P \cap Q = \emptyset$, then we may assume that Q is contained in B_1. Since $Q - t_1$ is incompressible in $B_1 - t_1$ and by Lemma 2.6, Q is isotopic $rel.t_1$ to P. This contradicts the hypothesis. Therefore $P \cap Q \neq \emptyset$.

We may assume that each component of $P \cap Q$ is a loop and $P \cap Q \cap L = \emptyset$, and that $|P \cap Q|$ is minimum among all 2-spheres ambient isotopic $rel.L$ to Q.

Claim 3.1. *Each component of $P \cap Q$ is a loop in P (in Q resp.) which splitts the four points $P \cap L$ ($Q \cap L$ resp.) into pairs of two points.*

Proof. Let l be an innermost component of $P \cap Q$ in P, and let D be the corresponding innermost disk in P. Here, by exchanging D if necessary, we may assume that $D \cap L$ consists of at most two points. If $D \cap L = \emptyset$, then by the incompressibility of $Q - L$ in $S^3 - L$ and the irreducibility of $B_i - t_i (i = 1, 2)$, we can reduce $|P \cap Q|$, and this contradicts the minimality of $|P \cap Q|$. If $D \cap L$ is one point, then by Lemma 2.2, we can reduce $|P \cap Q|$, and this contradicts the minimality of $|P \cap Q|$. This completes the proof. □

Claim 3.2. *$P \cap Q$ consists of a single loop.*

Proof. Suppose $P \cap Q$ consists of more than one loop. Then by Claim 3.1, those are mutually parallel in $P - L$ and in $Q - L$. Hence there is an annulus A in Q with $A \cap (P \cap Q) = \partial A$. Then A is an annulus properly embedded in B_1 or in B_2 which satisfies the hypothesis of Lemma 2.4. Thus we can reduce $|P \cap Q|$, and this contradicts the minimality of $|P \cap Q|$. This completes the proof because $P \cap Q \neq \emptyset$. □

Let D_1 and D_2 be the closure of the components of $Q-(P\cap Q)$ such that $D_1 \subset B_1$ and $D_2 \subset B_2$. Then for each $i = 1, 2$, D_i is a disk properly embedded in B_i which intersects t_i in two points. Further, by the incompressibility of $Q - L$ in $S^3 - L$ and the minimality of $|P \cap Q|$, it follows that D_i satisfies the hypothesis of Lemma 2.8. Therefore, (B_i, t_i) is a Montesinos tangle $T(\beta_i/\alpha_i, 1/2)$, where α_i and β_i are coprime integers and α_i is an odd integer greater than 1. Hence L is equivalent to a Montesinos link $M(0; (\alpha_1, \beta_1), (2, 1), (\alpha_2, \beta_2), (2, 1))$, where α_i and β_i are coprime integers and α_i is an odd integer greater than 1 $(i = 1, 2)$

Conversely, let L be the Montesinos link in Theorem 1.1, and let P and Q be the 2-spheres indicated in Figure 1. Then by Lemma 2.8, each of P and Q gives an essential free 2-string tangle decomposition of L.

Let Σ be the 2-fold branched covering space of S^3 along L. Then Σ is a Seifert fibered space over a 2-sphere with four singular fibers f_1, f_2, f_3 and f_4 such that the Seifert invariants of f_1, f_2, f_3 and f_4 are 1/2, 1/2, β_1/α_1 and β_2/α_2, where α_i is an odd integer greater than 1 $(i = 1, 2)$. Then we have;

$\pi_1(\Sigma) \cong \langle x_1, x_2, x_3, x_4, h | x_1^2 h = 1, x_2^2 h = 1, x_3^{\alpha_1} h^{\beta_1} = 1, x_4^{\alpha_2} h^{\beta_2} = 1, x_1 x_2 x_3 x_4 = 1, [x_i, h] = 1 (i = 1, 2, 3, 4)\rangle$.

Let P_0 and Q_0 be the preimages of P and Q in Σ by the covering projection. Then P_0 and Q_0 are incompressible tori saturated in the Seifert fibration.

Suppose that P and Q are mutually isotopic $rel.L$. Then P_0 and Q_0 are mutually isotopic in Σ, and hence the fiber f_3 is isotopic to the fiber f_4. This implies that $\alpha_1 = \alpha_2$ and that x_3 is conjugate to $x_4 h^b$ for some integer b.

Put $G = \pi_1(\Sigma)/\langle x_4, h\rangle \cong \langle x_1, x_2, x_3 | x_1^2 = 1, x_2^2 = 1, x_3^{\alpha_1} = 1, x_1 x_2 x_3 = 1\rangle$. Then by the above argument G is isomorphic to the group $H \cong \langle x_1, x_2 | x_1^2 = 1, x_2^2 = 1, x_1 x_2 = 1\rangle$ because $x_3 = w x_4 h^b w^{-1}$ and $x_4 = h = 1$. Then H is a cyclic group, and by Satz 3 of [6], G is not a cyclic group because of $\alpha_1 > 1$. This is a contradiction, and hence P and Q are not mutually isotopic.

The latter harf of Theorem 1.1 is contained in the proof of the former harf. This completes the proof of Theorem 1.1. □

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