Essential state surfaces for knots and links

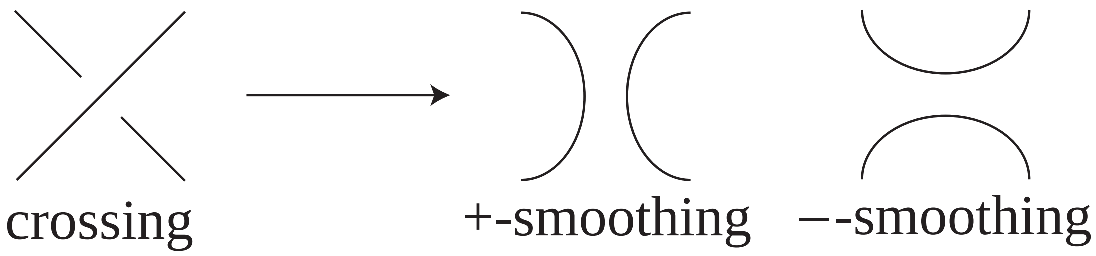
Makoto Ozawa
(Komazawa University)

November 2, 2007

K : a knot or link in S^3
D : a diagram of K on S^2

$\mathcal{C} = \{c_1, \ldots, c_n\}$: the set of crossings of D
$\sigma : \mathcal{C} \to \{+, -\}$: a state for D

For each crossing $c_i \in \mathcal{C}$, we take a $+$-smoothing or $-$-smoothing according to $\sigma(c_i) = +$ or $-$.

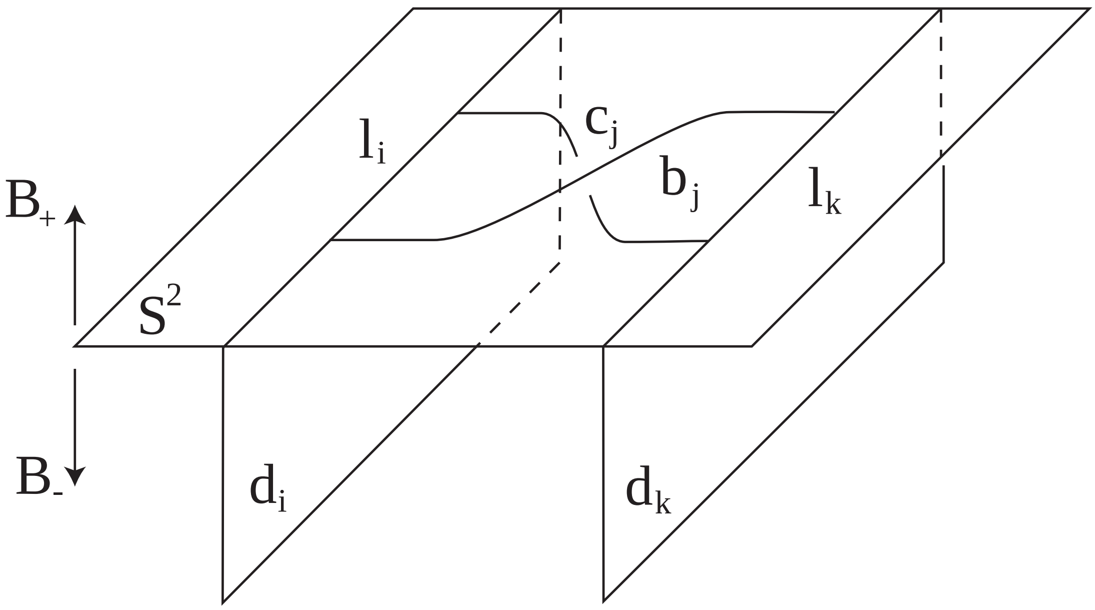


Then, we have a collection of loops $l_1, \ldots, l_m$ on S^2 and call those **state loops**.

$\mathcal{L}_\sigma = \{l_1, \ldots, l_m\}$: the set of state loops

The union of state loops $l_1 \cup \cdots \cup l_m$ bounds mutually disjoint disks $d_1 \cup \cdots \cup d_m$ in B_-.

We attach a half twisted band b_j to disks d_i, d_k so that it recovers the crossing c_j.



Then, we obtain a spanning surface which consists of disks $d_1, \ldots, d_m$ and half twisted bands $b_1, \ldots, b_n$ and call this a **σ-state surface** F_σ.

Historical remarks.

1. The state surfaces corresponding to the positive state σ_+ and negative state σ_- were considered for alternating links already in XIX century by Tait (and are called Tait surfaces).

2. The state surface corresponding to the Seifert state $\vec{\sigma}$, which gives the Seifert surface, was introduced by H. Seifert.

3. Independently, Prof. Jozef H. Przytycki had already thought about the concept of using a surface for any Kauffman state.

We construct a graph G_σ from F_σ by regarding a disk d_i as a vertex v_i and a band b_j as an edge e_j which has the same sign $\sigma(c_j)$.

We call the graph G_σ a σ-**state graph**.

Definition

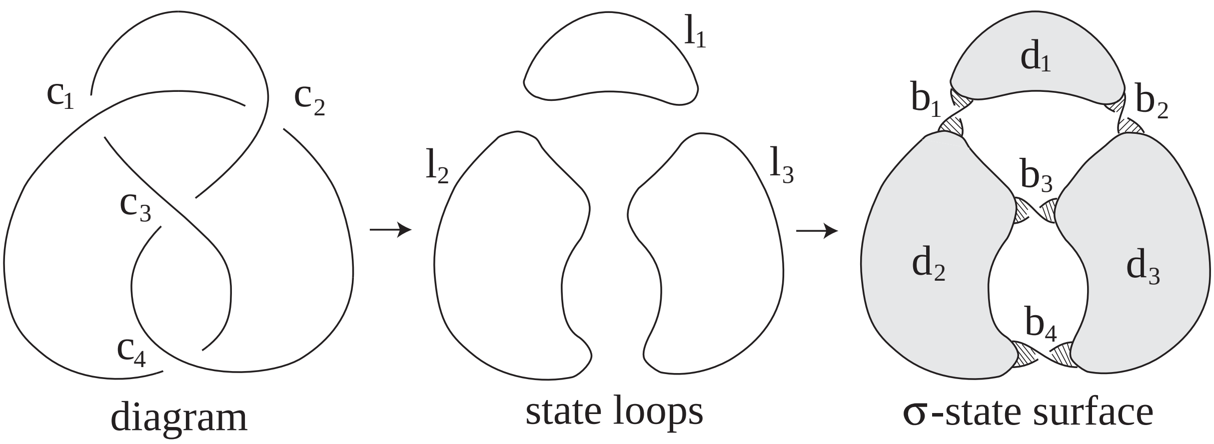
- D is σ-**adequate**
  $\Longleftrightarrow$ G_σ has no loop
- D is σ-**homogeneous**
  $\Longleftrightarrow$ in each block of G_σ, all edges have a same sign

Example 1.

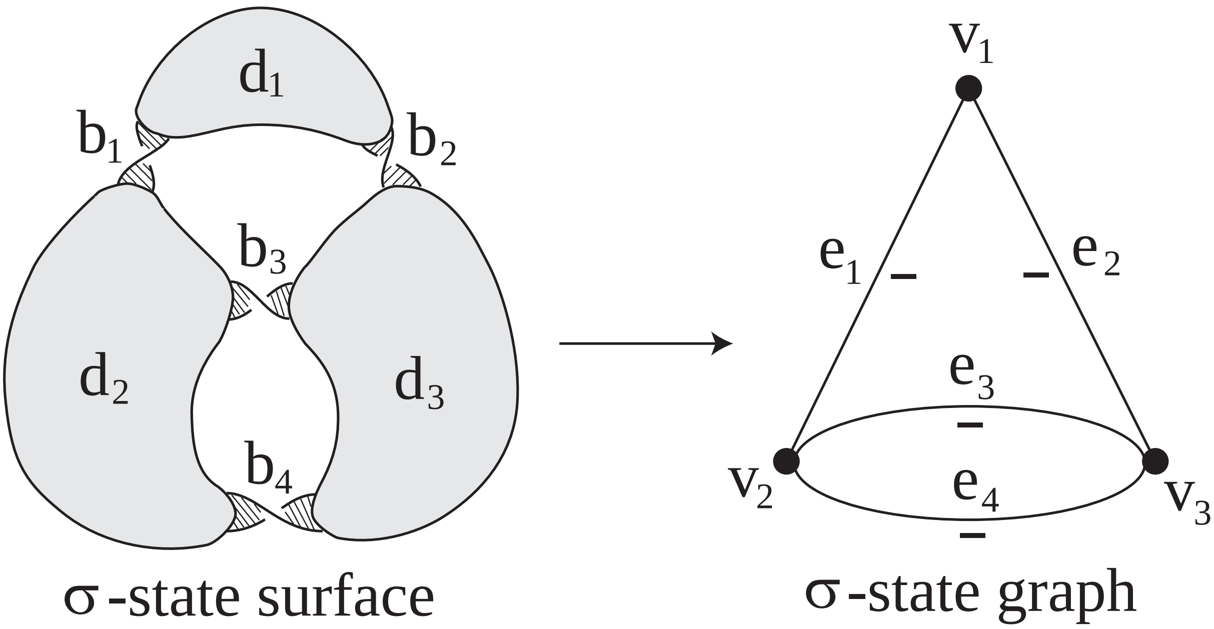
Let D be a diagram of the figure eight knot which has 4-crossings c_1, c_2, c_3, c_4 as Figure.

To make a σ-state surface, let σ be a negative state, i.e.

$$\sigma(c_1) = \sigma(c_2) = \sigma(c_3) = \sigma(c_4) = -$$



Since the σ-state graph G_σ has no loop and all edges have a $-$ sign, D is σ-adequate and σ-homogeneous.

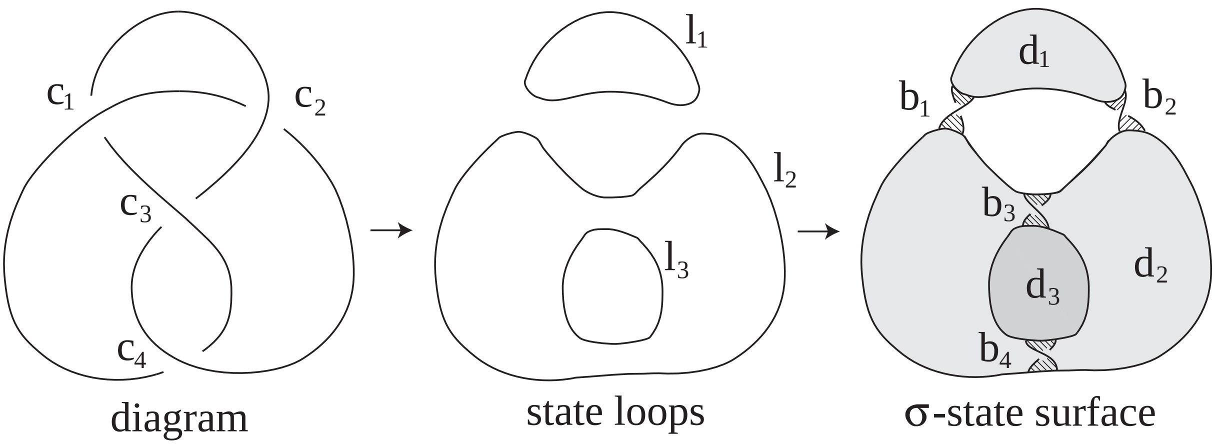


Example 2.

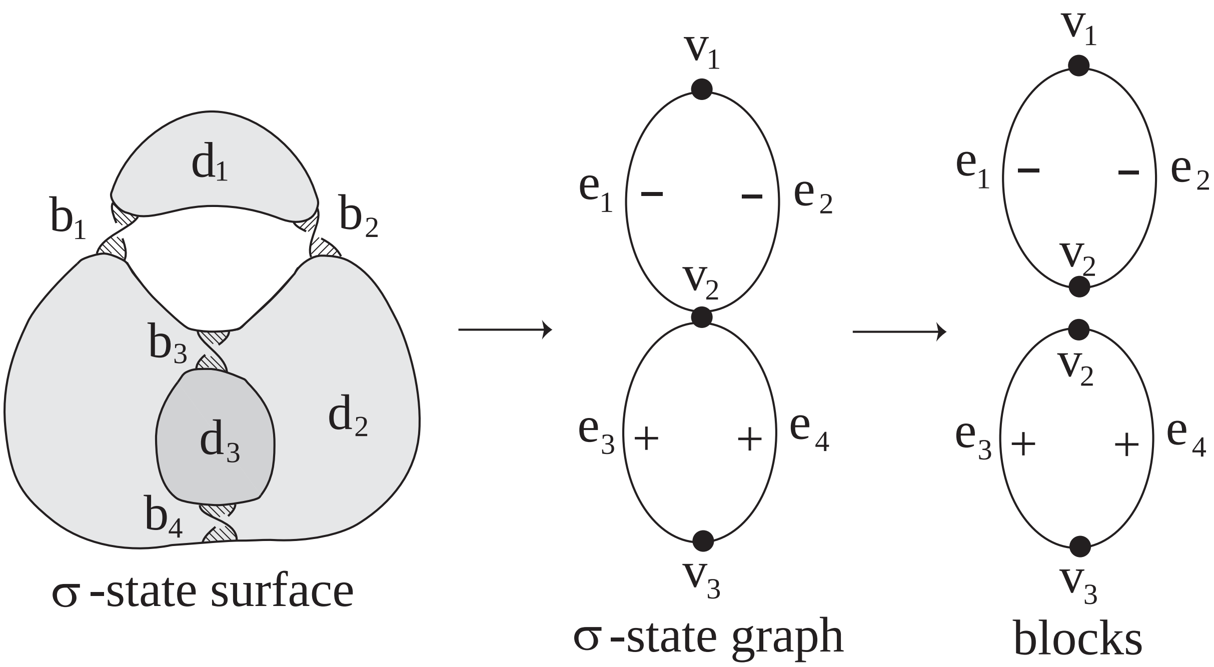
Let D be a diagram of the figure eight knot which has 4-crossings c_1, c_2, c_3, c_4 as Figure.

To make a σ-state surface, let σ be a Seifert state, i.e.

$\sigma(c_1) = \sigma(c_2) = -$ and $\sigma(c_3) = \sigma(c_4) = +$



Since the σ-state graph G_σ has no loop and all edges in each block have a same sign, D is σ-adequate and σ-homogeneous.



Example 3. Any positive diagram D is σ_+-adequate and σ_+-homogeneous for the positive state σ_+.

Example 4. Any alternating diagram D without nugatory crossings is $\sigma_\pm$-adequate and $\sigma_\pm$-homogeneous for the positive or negative state $\sigma_\pm$, and $\vec{\sigma}$-adequate and $\vec{\sigma}$-homogeneous for the Seifert state $\vec{\sigma}$.

Example 5. For any arborescent link L such that an absolute value of each weight is greater than 1, there exists a diagram D of L and a state σ such that D is σ-adequate and σ-homogeneous. Indeed, an arborescent link L is the boundary of a σ-state surface which is a Murasugi sum of twisted annuli or Möbius bands.

Example 6. By the definitions, homogeneous links (introduced by P. R. Cromwell), and semi-adequate links (introduced by W. B. R. Lickorish and M. B. Thistlethwaite) have σ-adequate and σ-homogeneous diagrams for some state σ.

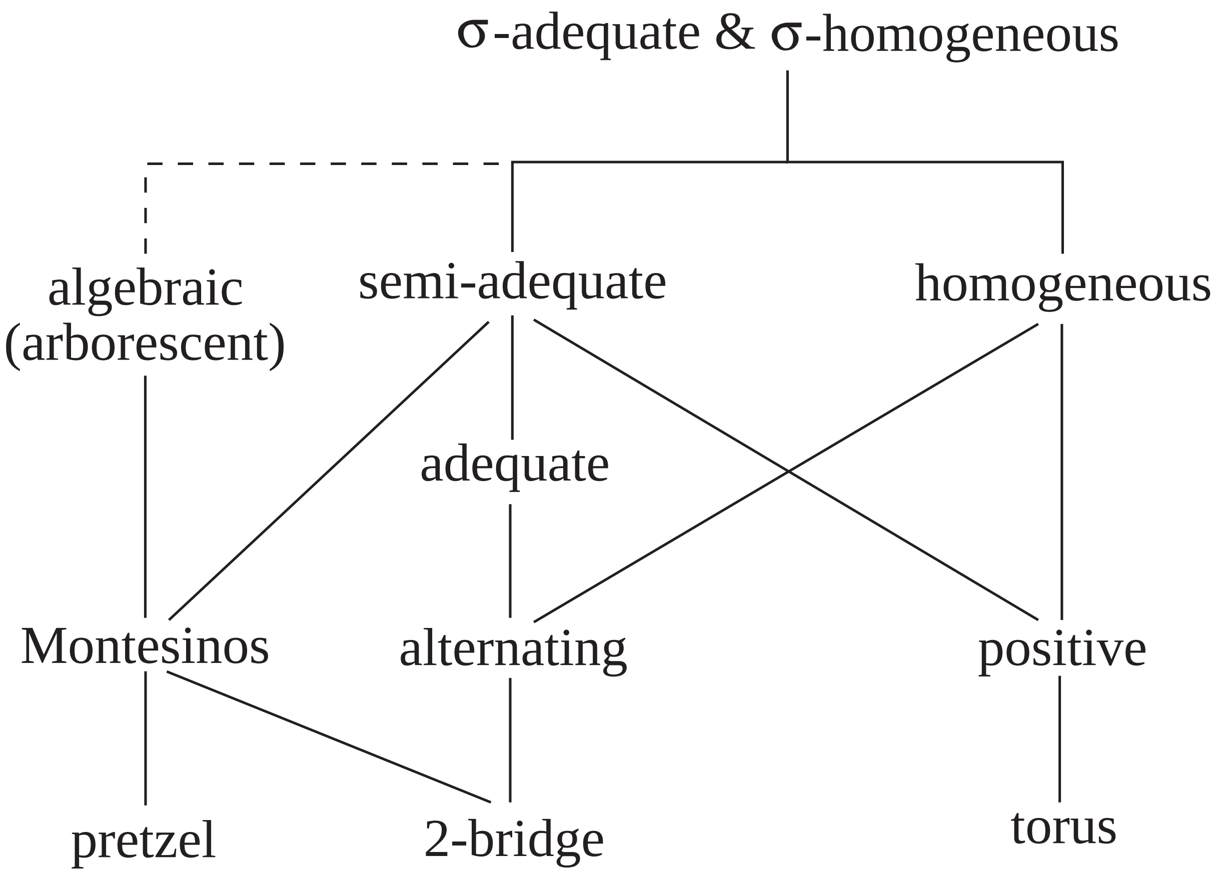
Here, a diagram D is **homogeneous** if in each block of $G_{\vec{\sigma}}$, all edges have a same sign. Since $F_{\vec{\sigma}}$ is a Seifert surface, a homogeneous diagram is automatically $\vec{\sigma}$-adequate.

Also, a diagram D is **semi-adequate** if G_{σ_+} or G_{σ_-} has no loop. In either case, a semi-adequate diagram D is automatically σ_+-homogeneous or σ_--homogeneous.

Incidentally, a diagram D is **adequate** if it is both σ_+-adequate and σ_--adequate.

The inclusion relations of various classes of knots and links are illustrated in Figure.

Here, the dotted line indicates that almost all algebraic links have σ-adequate and σ-homogeneous diagrams for some state σ, but some algebraic links seem to be not σ-adequate and σ-homogeneous for any state σ.



Theorem 1

If a diagram is σ-adequate and σ-homogeneous, then the σ-state surface is π_1-essential.

Theorem 2

Let K be a knot or link which admits a σ-adequate and σ-homogeneous diagram D without nugatory crossings. Then,

(1) D is non-trivial $\iff$ K is non-trivial.

(2) D is non-split $\iff$ K is non-split.

Lemma 1 (Aumann, 1956)

Let D be a reduced, prime, alternating diagram. Then, a checkerboard surface is π_1-essential.

This shows that subsurfaces F_i corresponding to each block of G_σ is π_1-essential.

Lemma 2

Let F_1 and F_2 be π_1-essential spanning surfaces. Then, $F_1 * F_2$ is also π_1-essential, where $*$ denotes the Murasugi sum (or $*$-product).

This shows that the σ-state surface F_σ is also π_1-essential, since it can be obtained from subsurfaces F_i by the Murasugi sums.

Problem.

1. Show that there exists a knot which has no σ-adequate and σ-homogeneous diagram. Furthermore, characterize the nature of knots and links which have σ-adequate and σ-homogeneous diagrams.

2. Determine primeness, satelliteness, fiberness, smallness and tangle decomposability from a given σ-adequate and σ-homogeneous diagram.

3. Show that for a given knot, the number of all σ-adequate and σ-homogeneous diagrams without nugatory crossings is finite.

4. Classify all knots and links which have σ-adequate and σ-homogeneous diagrams.