Rational structure on algebraic tangles and closed incompressible surfaces in the complements of algebraically alternating knots and links

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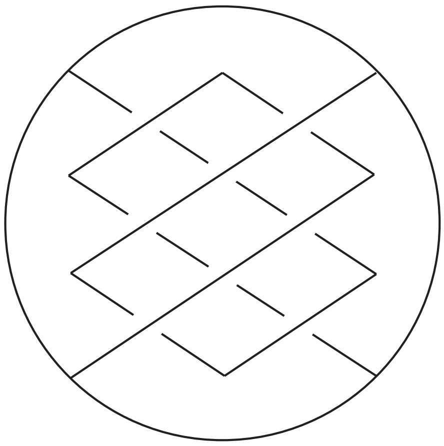
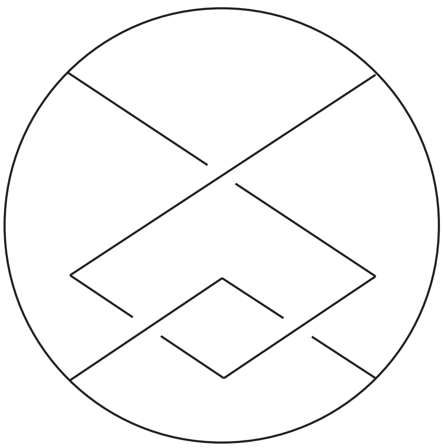
Conway

rational tangle $\longleftrightarrow$ rational number

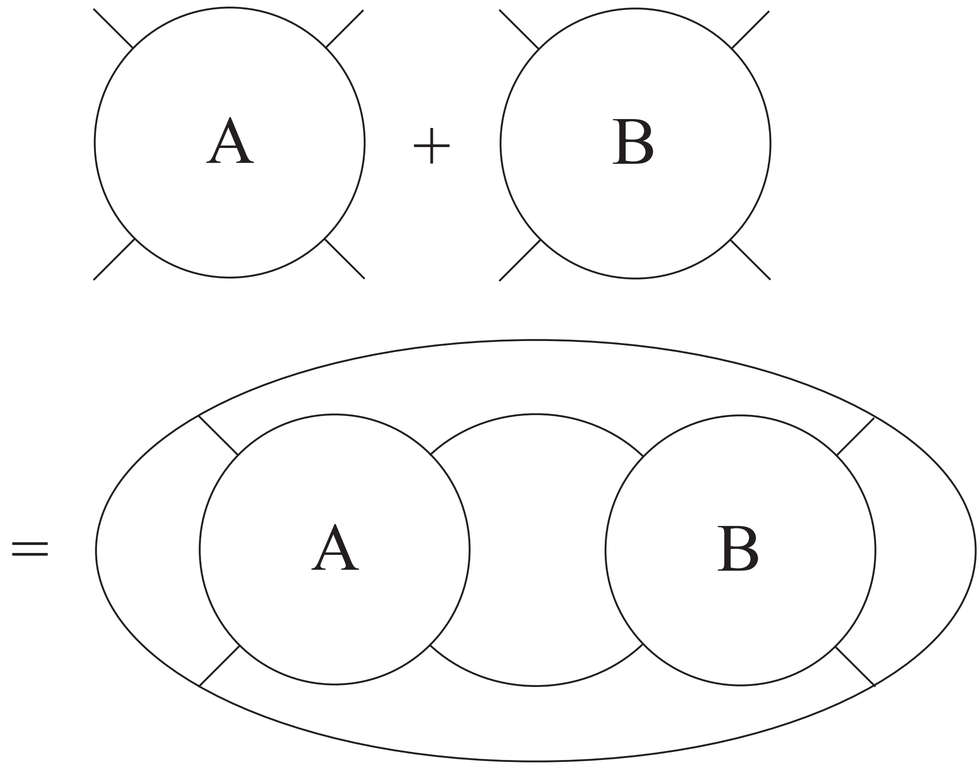


$\longleftrightarrow$ $\frac{2}{3}$

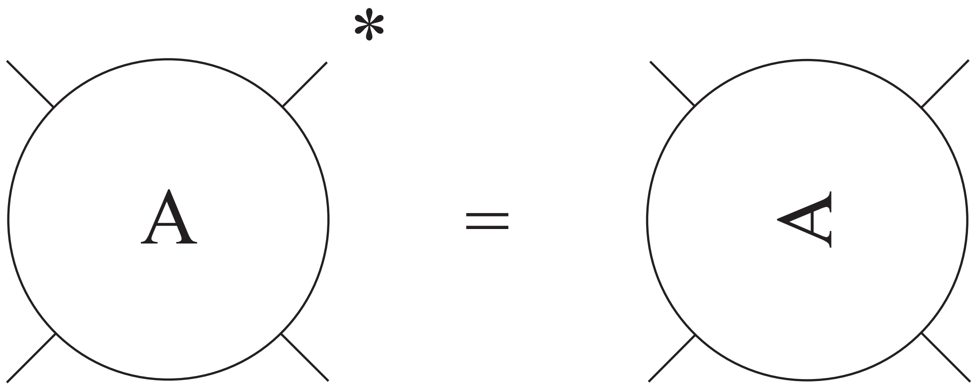


$\longleftrightarrow$ $\dfrac{1}{1+\dfrac{1}{2}}$

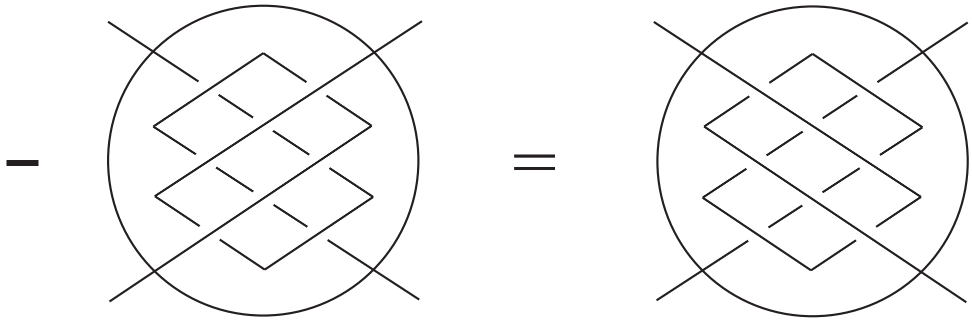
tangle sum



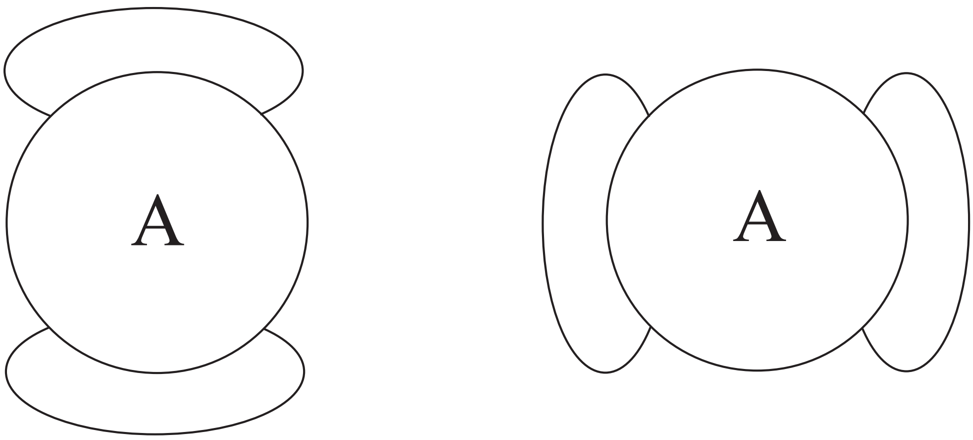
rotation



Reflection

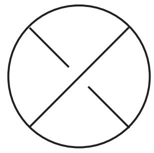
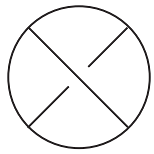


Numerator and denominator



N(T) D(T)

rational tangle

rational tangle = rational tangle +  or 

rational tangle = rational tangle*

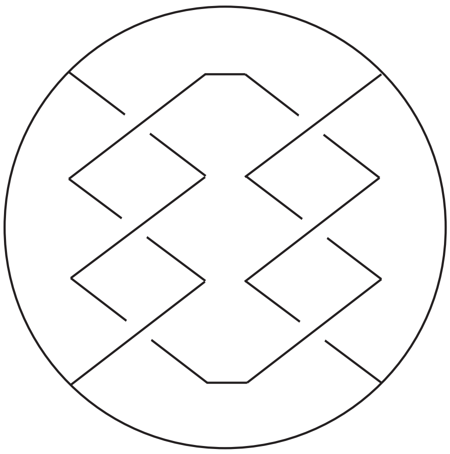
algebraic tangle

algebraic tangle = algebraic tangle + algebraic tangle

algebraic tangle = algebraic tangle*

Krebes

2-string tangle $\longrightarrow$ formal fraction



$\longrightarrow$ $$\frac{1}{3} + \frac{1}{3} = \frac{6}{9}$$

- $f(T_1 + T_2) = f(T_1) + f(T_2)$

- $f(-T) = -f(T)$

- $f(T^*) = -\frac{1}{f(T)}$

- $f(T) = p/q$
  $\Rightarrow |p| = \det N(T)$ and $|q| = \det D(T)$

Hereafter, we say that a surface is **essential** if it is

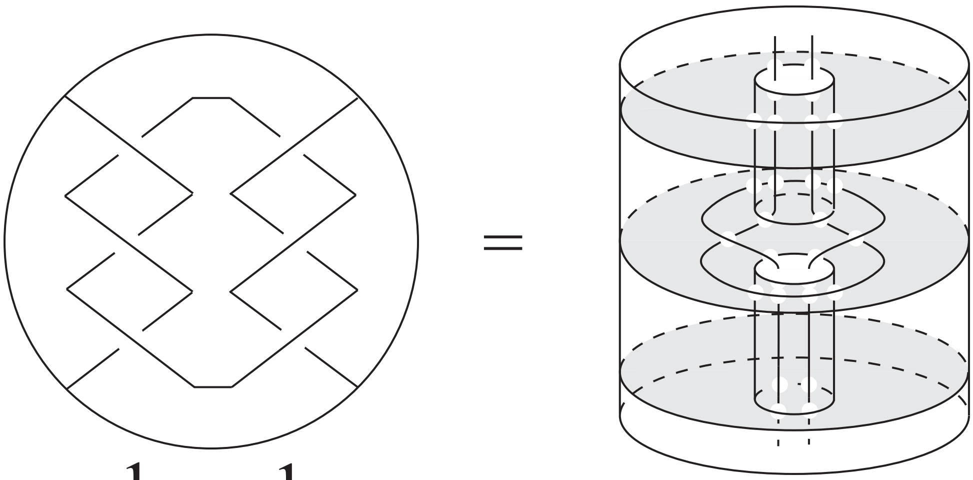
incompressible,
meridionally incompressible and
not boundary-parallel.

Theorem 1

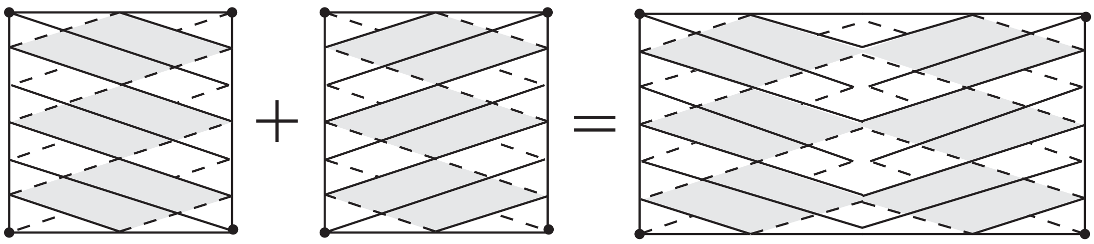
(B, T) : an algebraic tangle
$F \subset B - T$: an essential surface
$\Rightarrow$

- F separates the components of T in B
- the boundary slopes of essential surfaces in $B - T$ are unique

Example 1

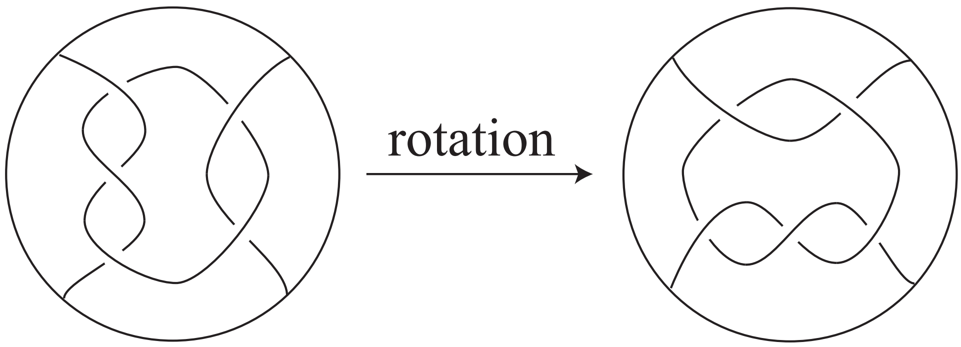


$$-\frac{1}{3}+\frac{1}{3} \qquad\qquad 0$$

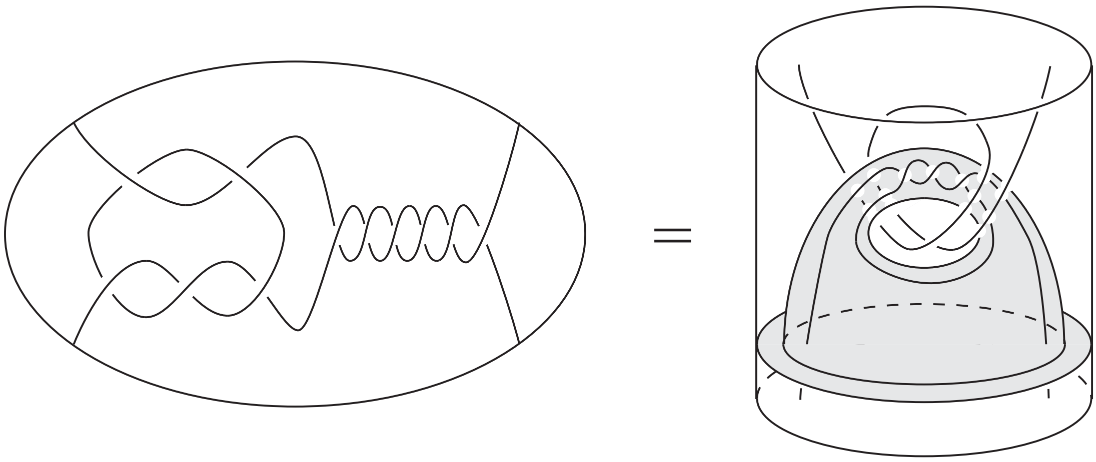


$$-\frac{1}{3} \qquad \frac{1}{3} \qquad 0$$

Example 2

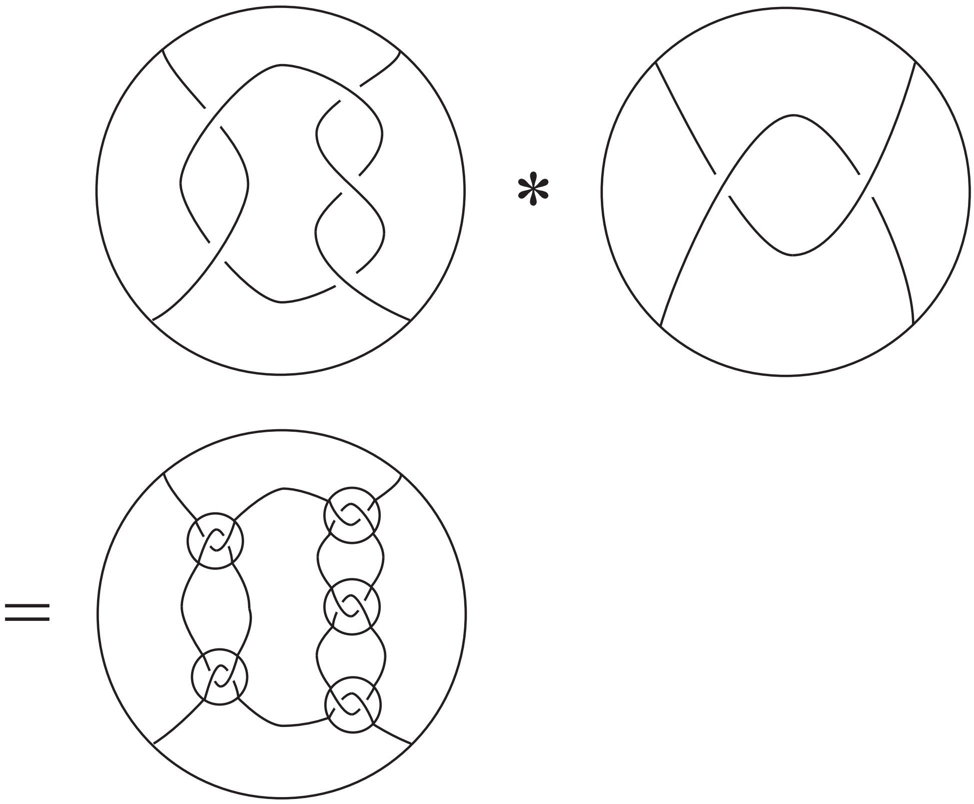


$$-\frac{1}{3}+\frac{1}{2}=\frac{1}{6} \qquad -6$$



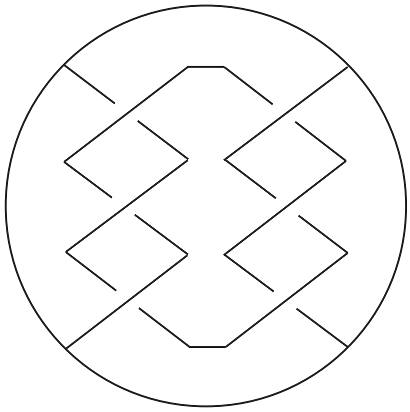
$$-6 + 6 \qquad 0$$

Multiplication



Theorem 2

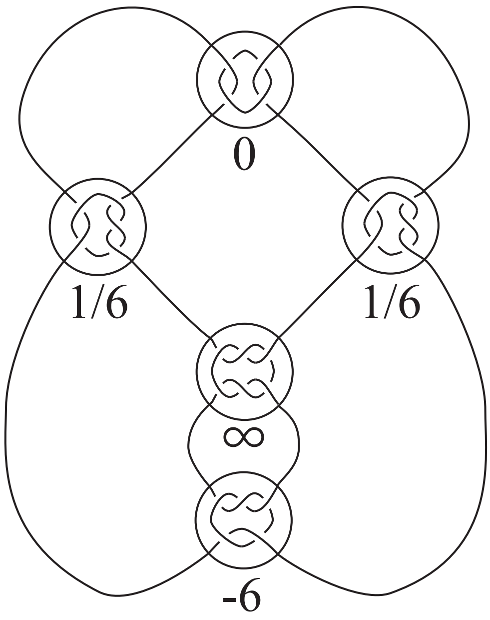
algebraic tangle $\longrightarrow$ boundary slope



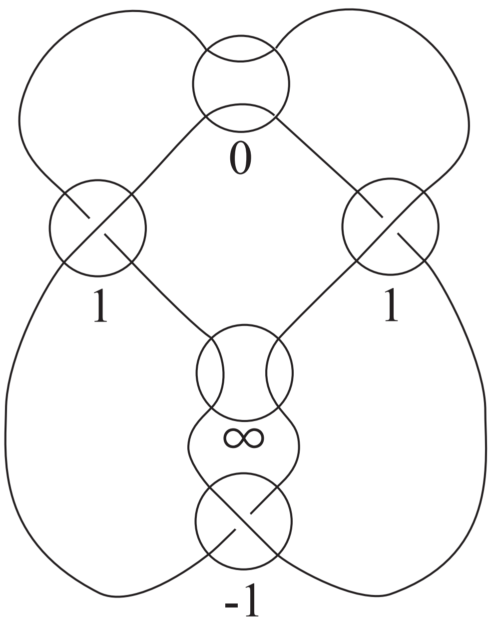
$\longrightarrow \frac{1}{3}+\frac{1}{3} = \frac{2}{3}$

- $\phi(T_1 + T_2) = \phi(T_1) + \phi(T_2)$
- $\phi(T_1 * T_2) = \phi(T_1)\phi(T_2)$
- $\phi(-T) = -\phi(T)$
- $\phi(T^*) = -\frac{1}{\phi(T)}$

Algebraically alternating link

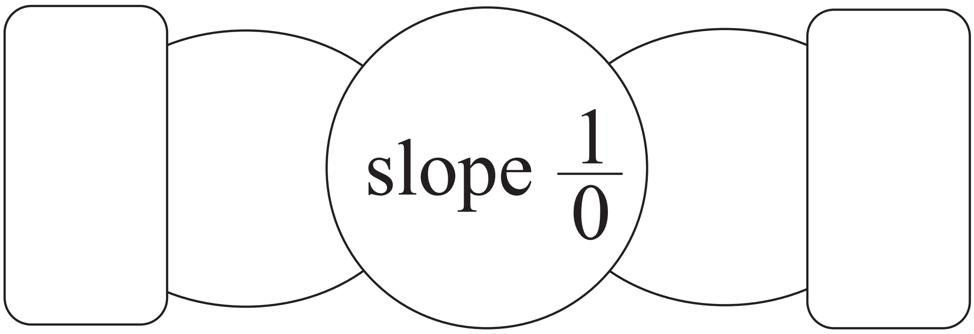


$\tilde{K}$: algebraically alternating link diagram

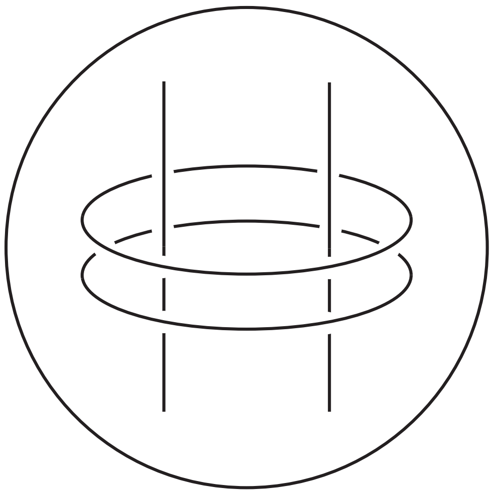


$\tilde{K}_0$: the basic diagram of $\tilde{K}$

cut tangle



Q_2



Q_2

Theorem 3

(S^3, K) : an algebraically alternating link
$F \subset S^3 - K$: an essential closed surface
$\Rightarrow$

- F separates the components of K in S^3
- $\tilde{K_0}$ is split or
  F is contained in an algebraic tangle of (S^3, K)
- If $F = S^2$, then there exists a cut tangle
- If $F = T^2$ and there exists no cut tangle, then (S^3, K) contains Q_2

Corollary

(S^3, K) : an algebraically alternating knot
$F \subset S^3 - K$: a closed incompressible surface
$\Rightarrow$ F is meridionally compressible

Proof of Theorem 1

Key Lemma 1

$(B,T) = (B_1,T_1) + (B_2,T_2)$

$\Rightarrow$ {essential surface in $B - T$}

$=$ {essential surface in $B_1 - T_1$}

$+$ {essential surface in $B_2 - T_2$}

Key Lemma 2

$(B,T) = (B_1,T_1) + (B_2,T_2)$

$F = F_1 + F_2$

F_1 separates T_1 in B_1

F_2 separates T_2 in B_2

$\Rightarrow$ F separates T in B

Key Lemma 3

(B,T) : rational tangle of slope p/q

$F \subset B - T$: essential surface

$\Rightarrow$ F is a disk separating T and its boundary slope is equal to p/q

Proof of Theorem 2

$\phi(T_1 + T_2) = \phi(T_1) + \phi(T_2)$ follows the proof of Key Lemma 1 and 2.

$\phi(-T) = -\phi(T)$ holds since a reflection changes the boundary slope -1 times.

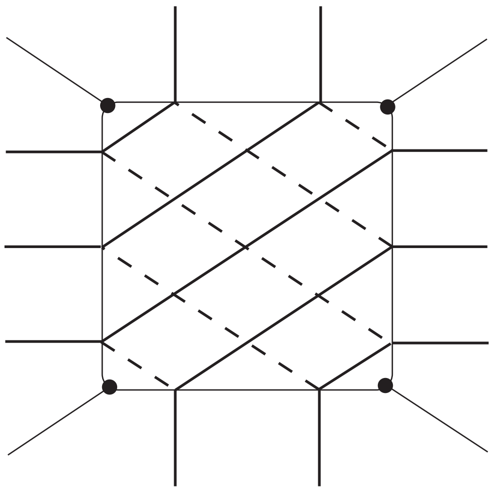
$\phi(T)\phi(T^*) = -1$ follows that a rotation changes the boundary slope reciprocally and times -1 by an orthogonal condition.

$\phi(T_1 * T_2) = \phi(T_1)\phi(T_2)$ is shown by an induction of the length of $T_1 = T_{11} + T_{12}$.

$$\begin{aligned}
\phi(T_1 * T_2) &= \phi((T_{11} + T_{12}) * T_2) \\
&= \phi(T_{11} * T_2 + T_{12} * T_2) \\
&= \phi(T_{11} * T_2) + \phi(T_{12} * T_2) \\
&= \phi(T_{11})\phi(T_2) + \phi(T_{12})\phi(T_2) \\
&= (\phi(T_{11}) + \phi(T_{12}))\phi(T_2) \\
&= \phi(T_{11} + T_{12})\phi(T_2) \\
&= \phi(T_1)\phi(T_2)
\end{aligned}$$

Proof of Theorem 3

In the algebraically alternating diagram $\tilde{K}$, we put an essential surface F in a "standard position".



an intersection with an algebraic tangle of slope 2/3

By the alternating property of the basic diagram $\tilde{K_0}$, a Menasco-like argument works.

In consequence, it follows that F intersects algebraic tangles only in one of slope 0 or ∞.

This shows that $\tilde{K_0}$ is split.